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A new primitive variable recovery scheme in GRMHD code Spritz

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W. Kastaun, J.V. Kalinani, R. Ciolfi 2021, PRD 103-023018, ArXiv:2005.01821
J.V. Kalinani et al. - in preparation

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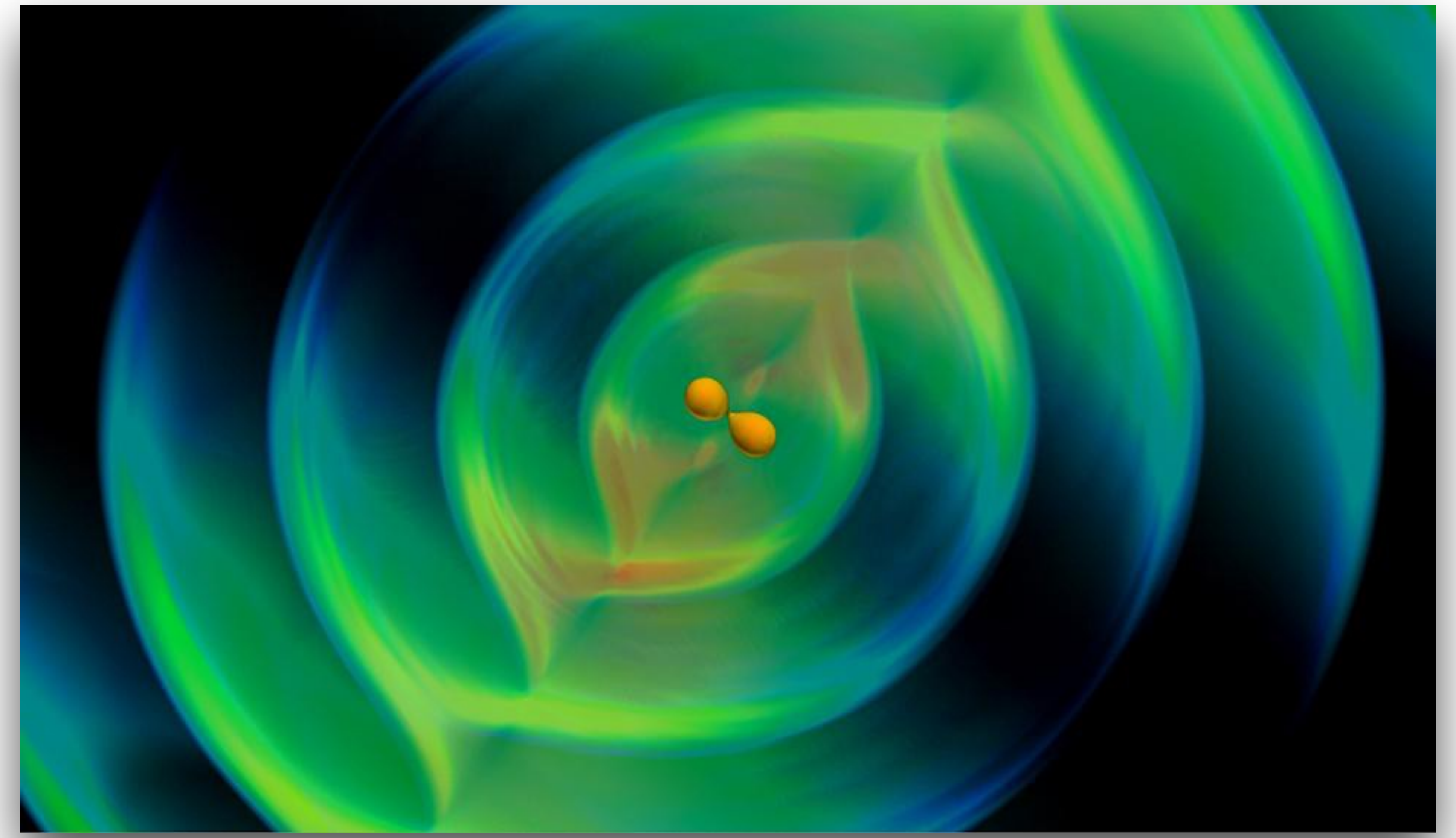
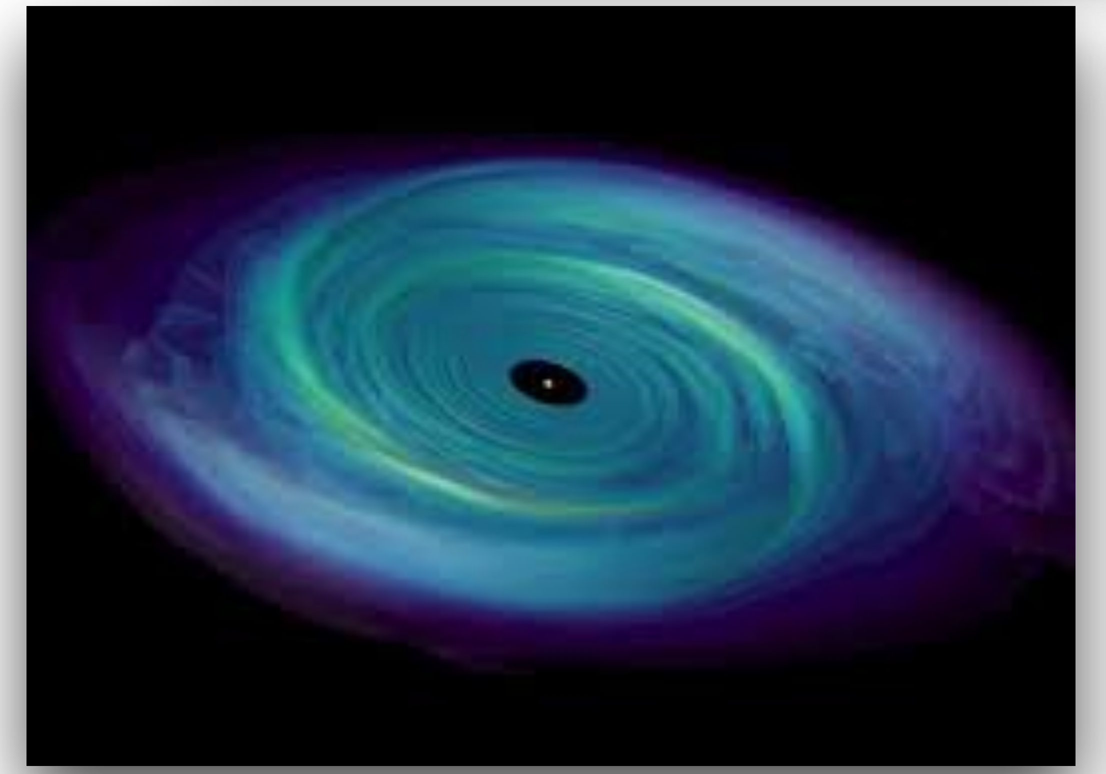
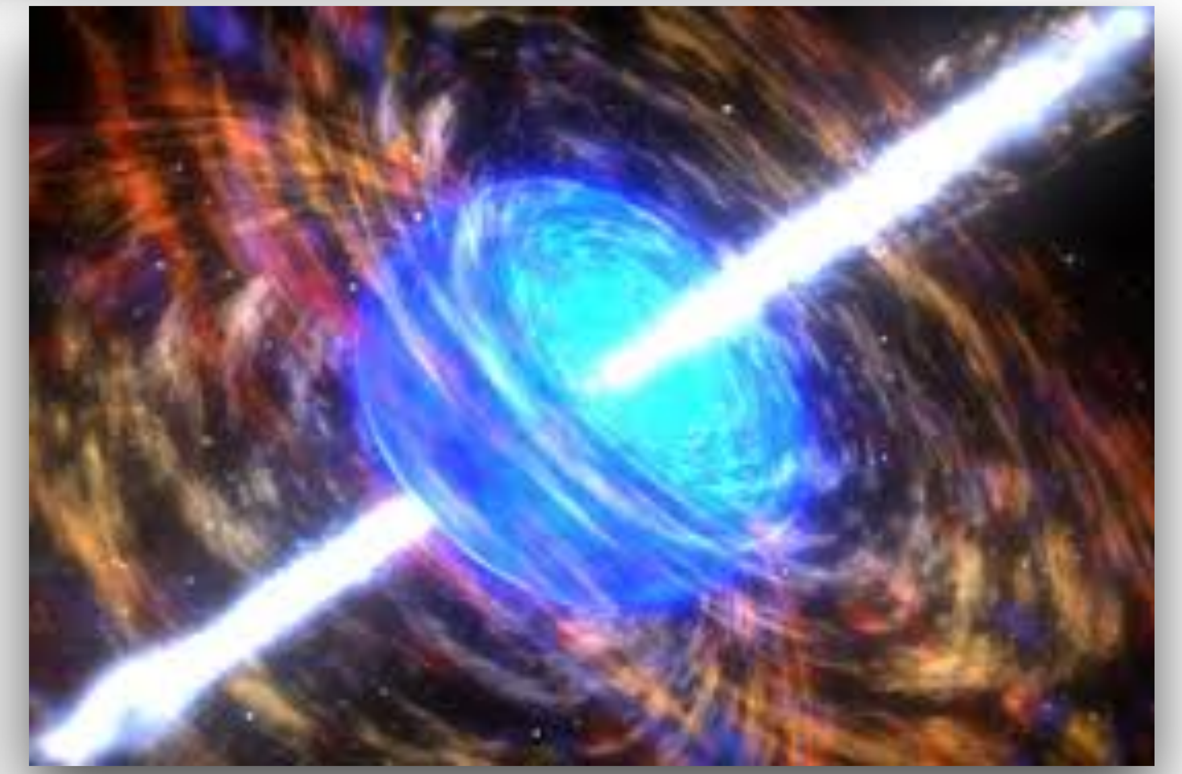
General Relativistic Magnetohydrodynamics

- Fundamental tool to investigate various astrophysical phenomena/events:
  - BNS/NS-BH mergers
  - relativistic jets
  - core-collapse supernovae
  - accretion disks

- Recently developed Spritz GRMHD code:
  - Cipolletta+2020, Cipolletta+2021
  - based on flux-conservative Valencia formulation
  - vector potential staggered evolution
  - designed to work within Einstein Toolkit framework







GRMHD Equations

Flux-conservative formalism

$$\frac{1}{\alpha\sqrt{\gamma}}\left[\partial_t\left(\sqrt{\gamma}\mathbf{U}\right)+\partial_i\left(\sqrt{\gamma}\mathbf{F}^{\mathbf{i}}\right)\right]=\mathbf{S}$$

State vector

$$\mathbf{U}\equiv\begin{pmatrix} D \\ S_j \\ \tau \\ B^k \\ DY_e \end{pmatrix}$$

Fluxes

$$\mathbf{F}\equiv\begin{pmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i+\alpha\left(P+P_{\mathrm{mag}}\right)\delta^i_j-\alpha b_j B^i/W \\ \tau\tilde{v}^i+\alpha\left(P+P_{\mathrm{mag}}\right)v^i-\alpha^2 b^0 B^i/W \\ B^k\tilde{v}^i-B^i\tilde{v}^k \\ DY_e\tilde{v}^i \end{pmatrix}$$

Source terms

$$\mathbf{S}\equiv\begin{pmatrix} 0 \\ T^{\mu\nu}\left(\partial_\mu g_{\nu j}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\partial_\mu\ln\alpha-T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0 \\ 0 \end{pmatrix}$$

Conservative and Primitive Variables

conservatives

conserved density	$D \equiv \rho W$
conserved momentum	$S_j \equiv \left(\rho h + b^2\right) W^2 v_j - \alpha b^0 b_j$
conserved internal energy	$\tau \equiv \left(\rho h + b^2\right) W^2 - \left(P + P_{\rm mag}\right) - \alpha^2 \left(b^0\right)^2 - D$
conserved magnetic field	$\vec{B} \equiv \vec{\nabla} \times \vec{A}$
conserved electron fraction	DY_e

primitives

ρ	rest-mass density
v^i	fluid 3-velocity
ϵ	specific internal energy
Y_e	electron fraction

P	gas pressure
$P_{\rm mag} \equiv b^2/2$	magnetic pressure
$W = 1/\sqrt{1 - v^2}$	Lorentz factor
$h = 1 + \epsilon + P/\rho$	specific enthalpy

Equation of State

- **Analytical**

- ideal gas $P(\rho, \epsilon) = (\Gamma - 1)\rho\epsilon; \;\; \epsilon_{\rm min} = 0$

- polytrope $P(\rho) = k\rho^{\Gamma}; \quad \Gamma = 1 + 1/n$

- **Tabulated**

- 1 parameter $P = P(\rho)$

- 3 parameter $P = P(\rho, T, Y_e)$

- **Hybrid**

$$P(\rho, \epsilon) = P_{\rm cold}(\rho) + (\Gamma_{\rm th} - 1)\rho\left(\epsilon - \epsilon_{\rm cold}(\rho)\right)$$

$$\epsilon_{\rm min}(\rho) = \epsilon_{\rm cold}(\rho)$$

- Causality & stability: $0 \leq c_s^2 \leq 1$
- Dominant energy condition: $P \leq \rho_{\rm E}$

Latest version of Spritz provides support for analytical, hybrid as well as tabulated EOSs

Recovery Problem

- Recover primitive variables from evolved conservative ones: C2P
  - no analytical solution in GRMHD
  - needs numerical approach: root finding algorithms

- Currently known C2P schemes (e.g., Noble+2006, Duran+2008, Neilsen+2014, Newman+2014, Siegel+2018) could all fail in certain regimes, e.g. for high magnetizations and Lorentz factors

Types of methods:

- Newton-Raphson schemes
  - unbounded; usually need initial guess; depend on EOS derivatives

- Root-bracketing schemes
  - bounded; might not depend on initial guess and on EOS derivatives

Kastaun C2P Scheme [Kastaun+2021]

- Uses root-bracketing scheme
- Alway converges to a unique solution (mathematical proof)
- Strong error policy: guarantees to find invalid evolved variables and applies harmless corrections, if necessary
- EOS-agnostic
- Publicly available code along with an EOS-framework on Zenodo: wokast/RePrimAnd

see Wolfgang Kastaun's talk on Feb 26th

Master function:

- Requirements
  - one dimensional & continuous
  - always has exactly one root
  - known interval
  - no EOS derivatives
  - well behaved in Newtonian regime

independent variable

$$\boxed{\mu \equiv \frac{1}{Wh}}$$

master function

$$\boxed{\begin{aligned} f(\mu) &= \mu - \hat{\mu}(\mu) \\ \hat{\mu}(\mu) &= \frac{1}{\hat{\nu}(\mu) + \bar{r}^2(\mu)\mu} \end{aligned}}$$

Kastaun C2P Scheme

- Analytical error bounds provided
- Error propagation of the root finding accuracy used to quantify the accuracy of the recovered primitives
- Single free parameter Δ controls the accuracy in a well-defined way

$$\frac{\delta\hat{W}}{\hat{W}} \leq \hat{v}^2\Delta\,, \qquad \Delta \equiv \hat{W}^2\frac{\delta\mu}{\mu}$$

$$\frac{\delta\hat{z}}{\hat{z}} \leq \Delta\,, \qquad \frac{\delta\hat{v}}{\hat{v}} \leq \frac{|\delta\hat{v}^i|}{\hat{v}} \leq \frac{\Delta}{\hat{W}^2}$$

$$\frac{\delta\hat{\rho}}{\hat{\rho}} \leq \hat{v}^2\Delta\,, \qquad \frac{\delta\hat{h}}{\hat{h}} \leq \hat{v}^2\Delta$$

$$\frac{\delta\hat{\epsilon}}{1+\hat{\epsilon}} \leq \hat{a}\hat{v}^2\Delta\,, \qquad \frac{\delta\hat{\epsilon}}{\hat{\epsilon}} \leq \left(1+\hat{\epsilon}\right)\frac{\hat{a}}{\hat{\epsilon}}\hat{v}^2\Delta$$

$$\frac{\delta\hat{\rho}_E}{\hat{\rho}_E} \leq 2\Delta\,, \qquad \frac{\delta\hat{P}}{\hat{P}} \leq \hat{v}^2\left(1+\hat{a}\right)\frac{\hat{c}_s^2}{\hat{a}}\Delta$$

Kastaun C2P Scheme: Test results

- *Left:* Relative error of reconstructed pressure; demanded accuracy, $\Delta = 10^{-8}$
- *Right:* Number of EOS calls to reconstruct the primitives; demanded accuracy, $\Delta = 10^{-8}$

Validated for high magnetisations $\left(B = 10^{17}\,\mathrm{G}\sqrt{\dfrac{\rho}{10^{12}\ \mathrm{g/cm^3}}}\right)$ and highly relativistic regimes (W=1000)

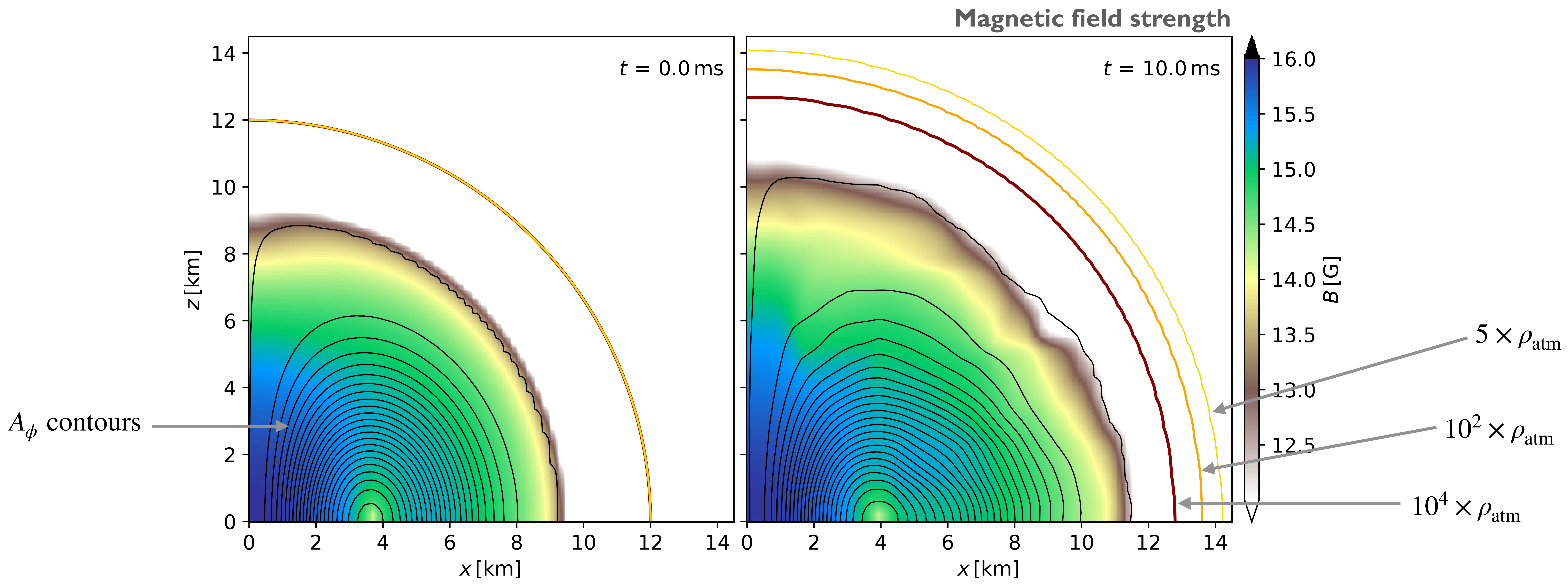
Kastaun C2P Scheme in Spritz

- Public version of Spritz equipped with Anton+2006, Noble+2006, Palenzuela+2015 C2P
- Integrated RePrimAnd library into Einstein Toolkit
- Added option to use C2P from RePrimAnd
- Defines and enforces validity range for EOS
- Option to use different error policy within BHs
- Support for fully tabulated EOS underway

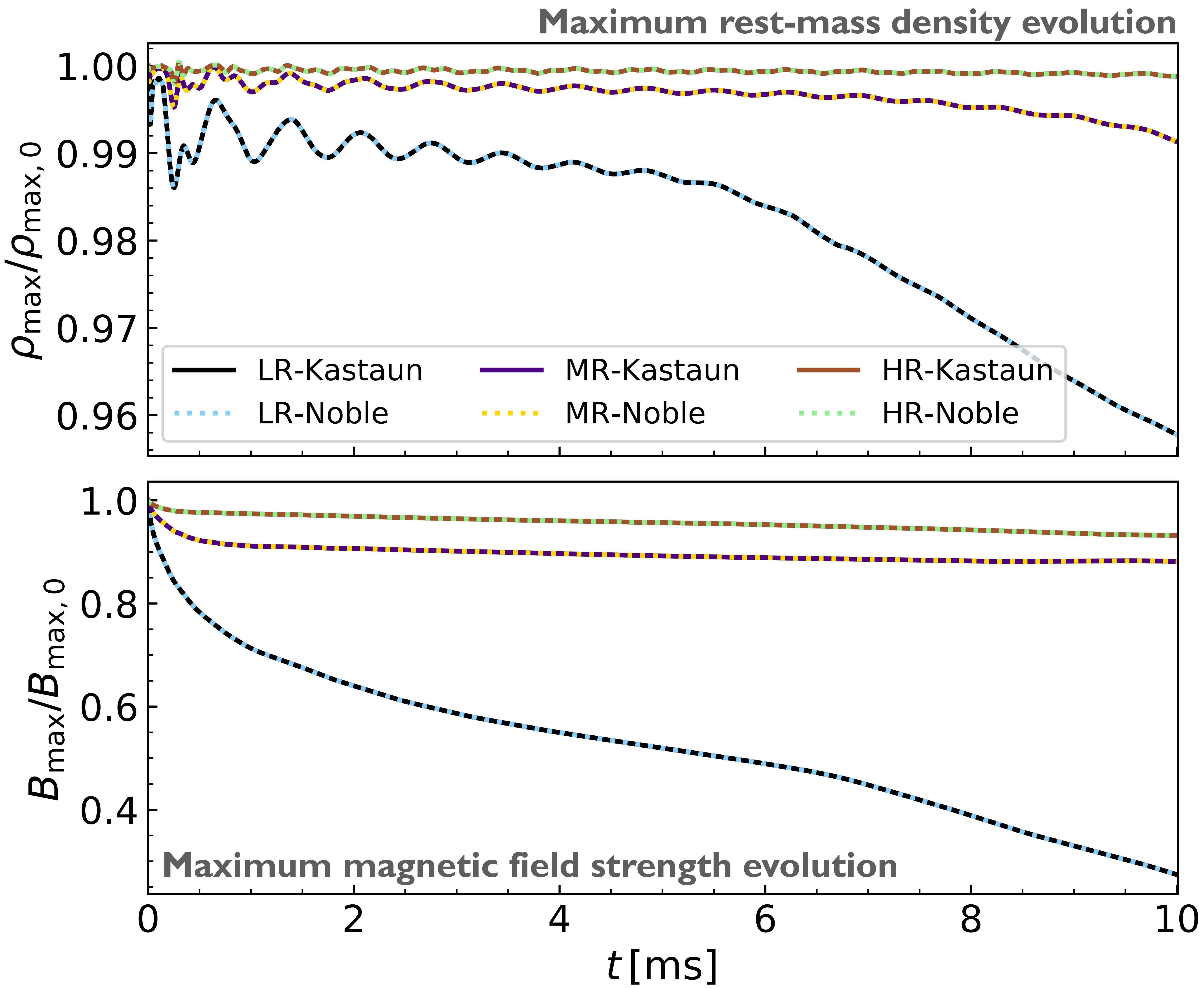
I. Neutron star with internal magnetic field

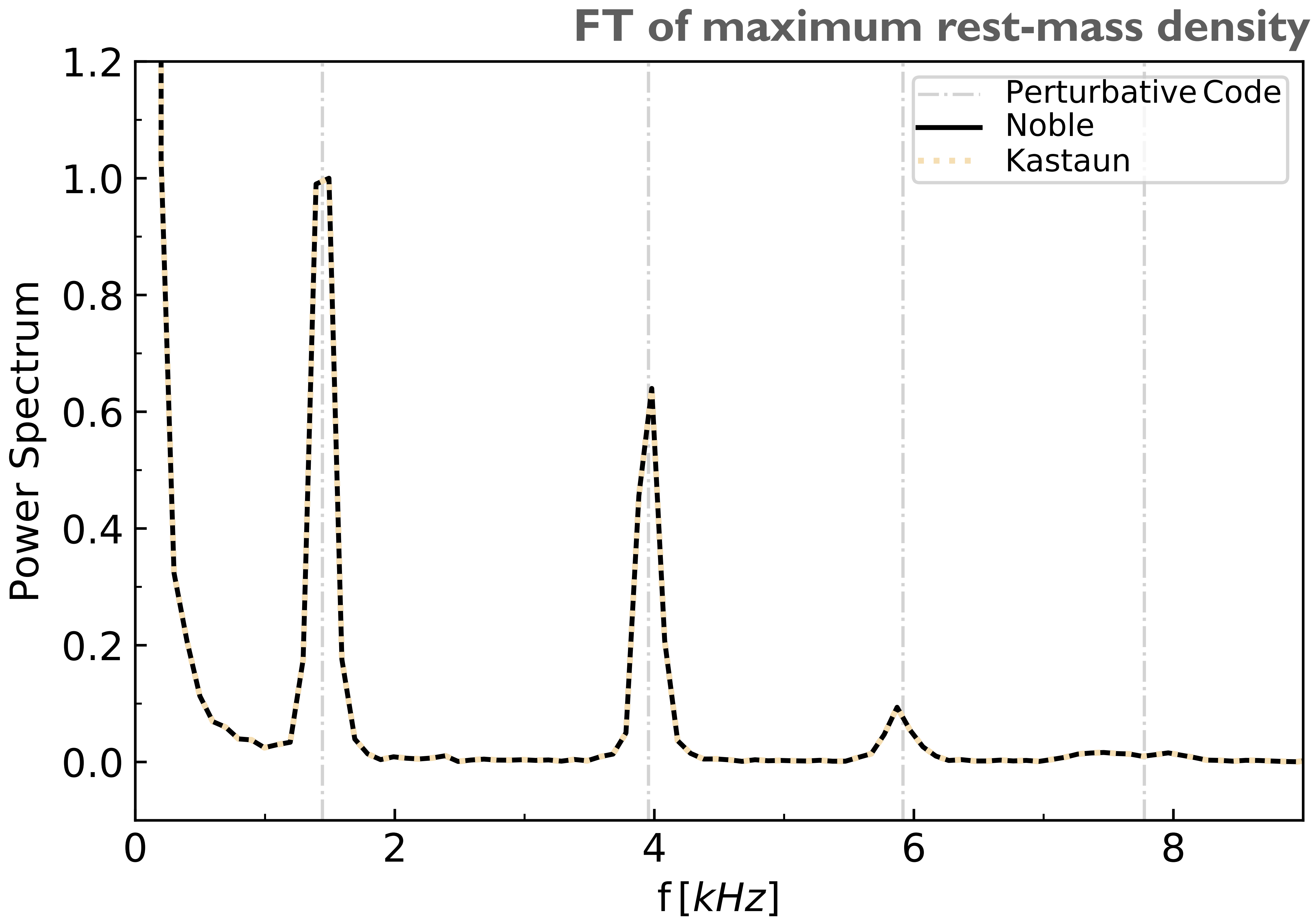
- Initial data built via TOVSolver with a polytropic EOS ($\Gamma = 2,\ K = 100$) EOS; evolved with IG EOS

$$\rho_{\mathrm{c,initial}} = 8.06 \times 10^{14}\,\mathrm{g/cm^3}; \quad \rho_{\mathrm{atm}} = 6.3 \times 10^{6}\,\mathrm{g/cm^3}; \quad B_{\mathrm{max}} = 1 \times 10^{16}\,\mathrm{G}$$



I. Neutron star with internal magnetic field





Exact matching curves for Kastaun and Noble C2P

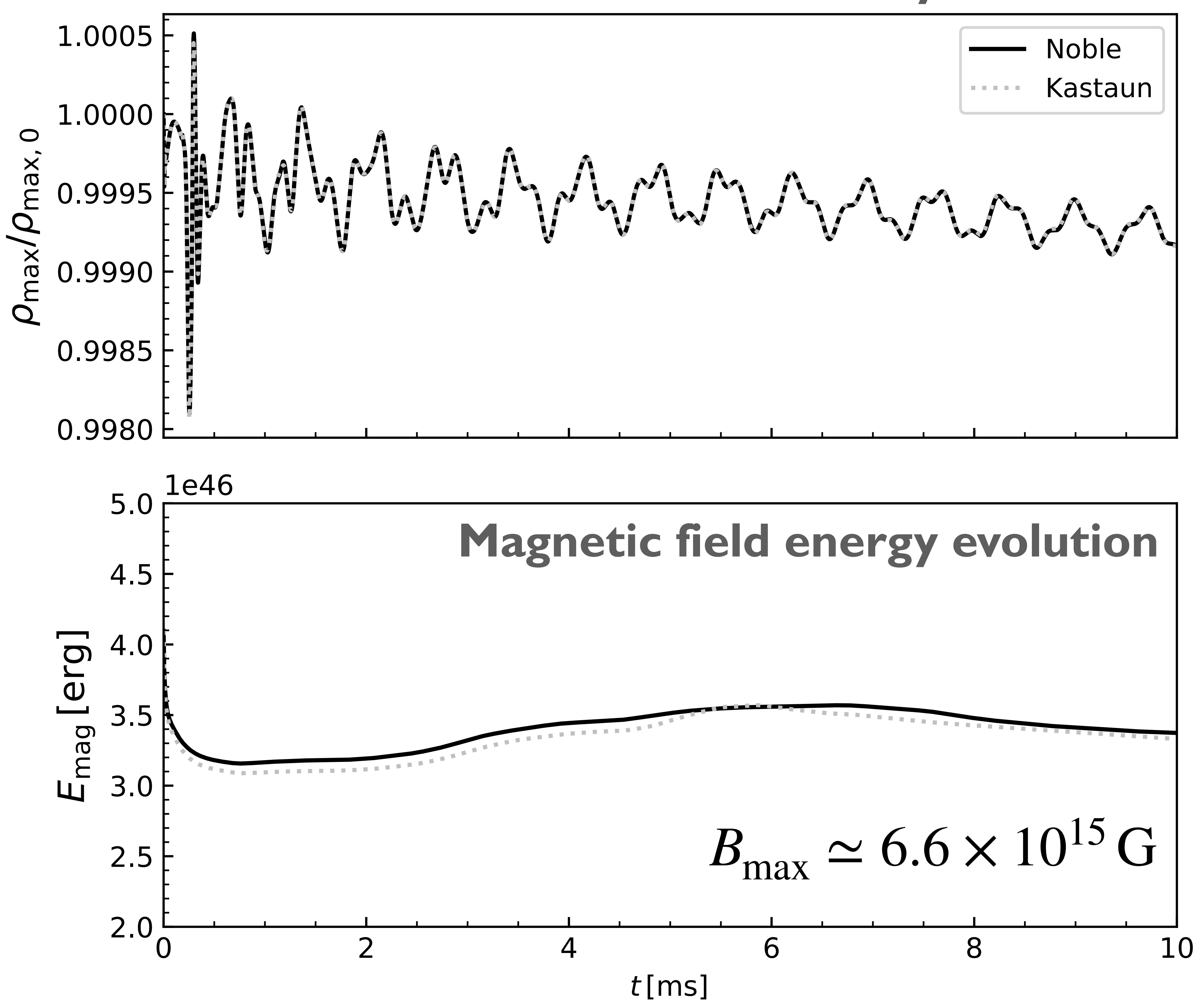
II. NS with extended dipolar field

- Vector potential: **Moesta+2020**

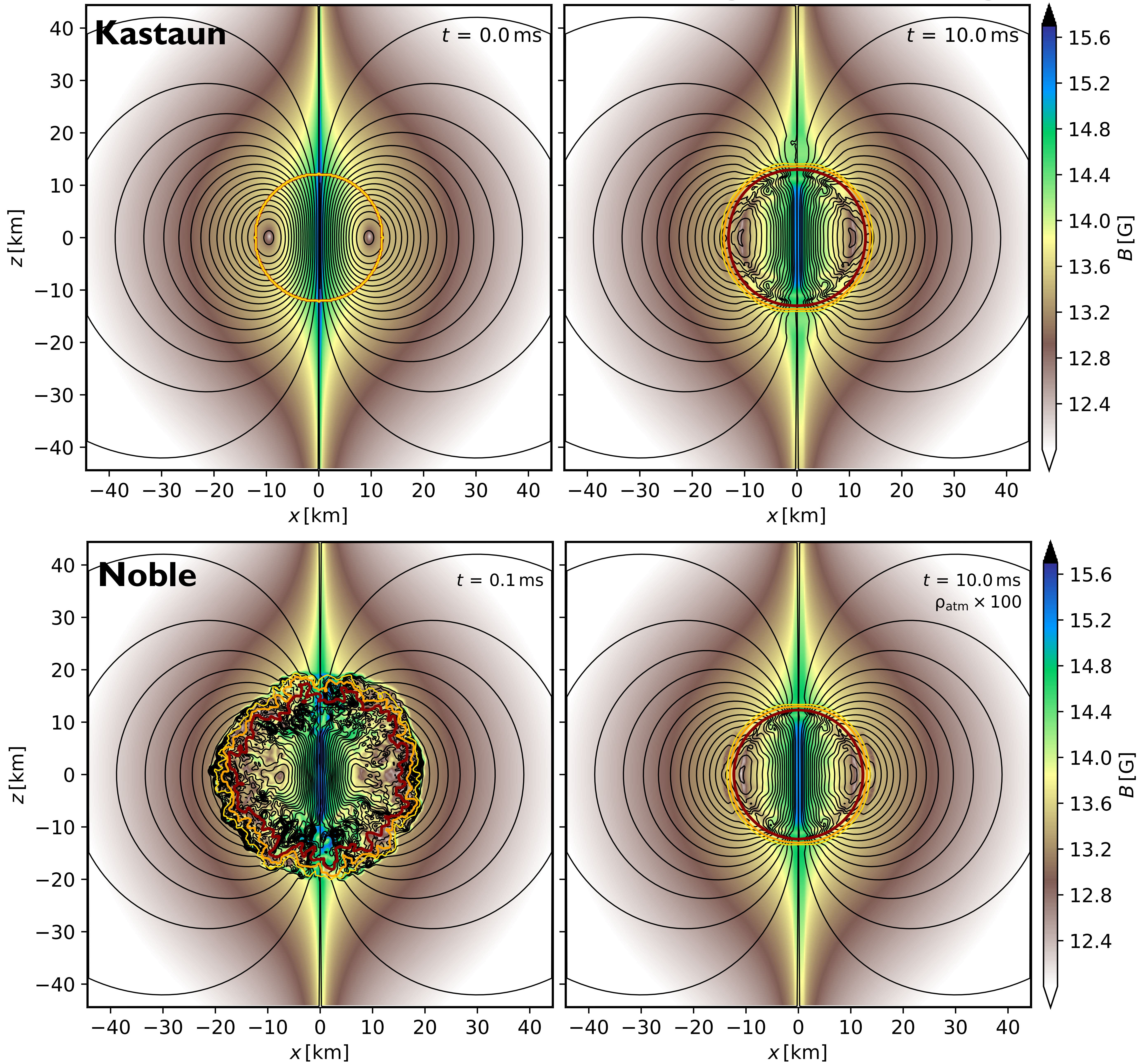
$$A_\phi = B_0(r_0^3)(r^3 + r_0^3)^{-1} r \sin\theta$$

- Floor density: $\rho_{\rm atm} = 6.3 \times 10^5\ {\rm g/cm^3}$

Max rest-mass density evolution



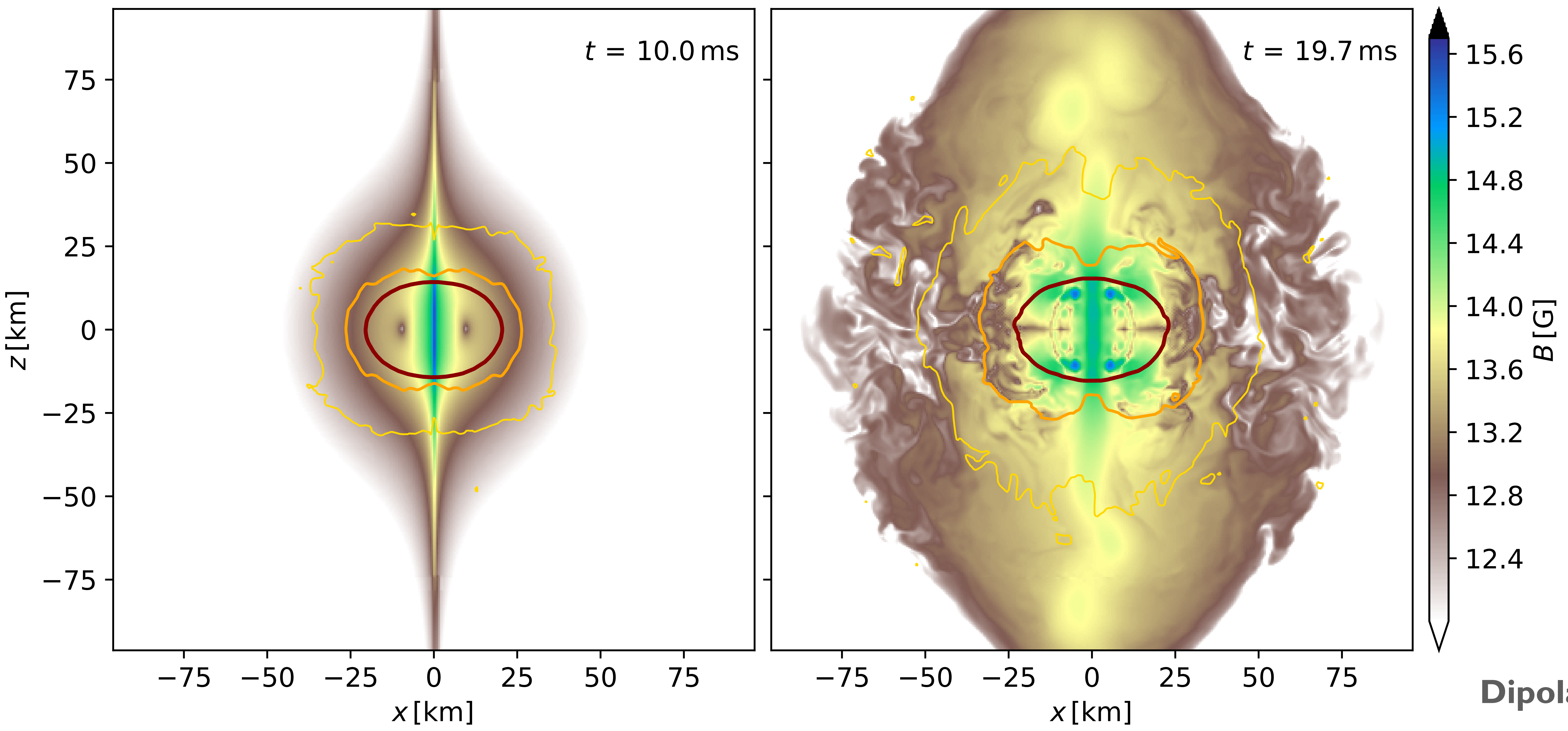
Magnetic field strength



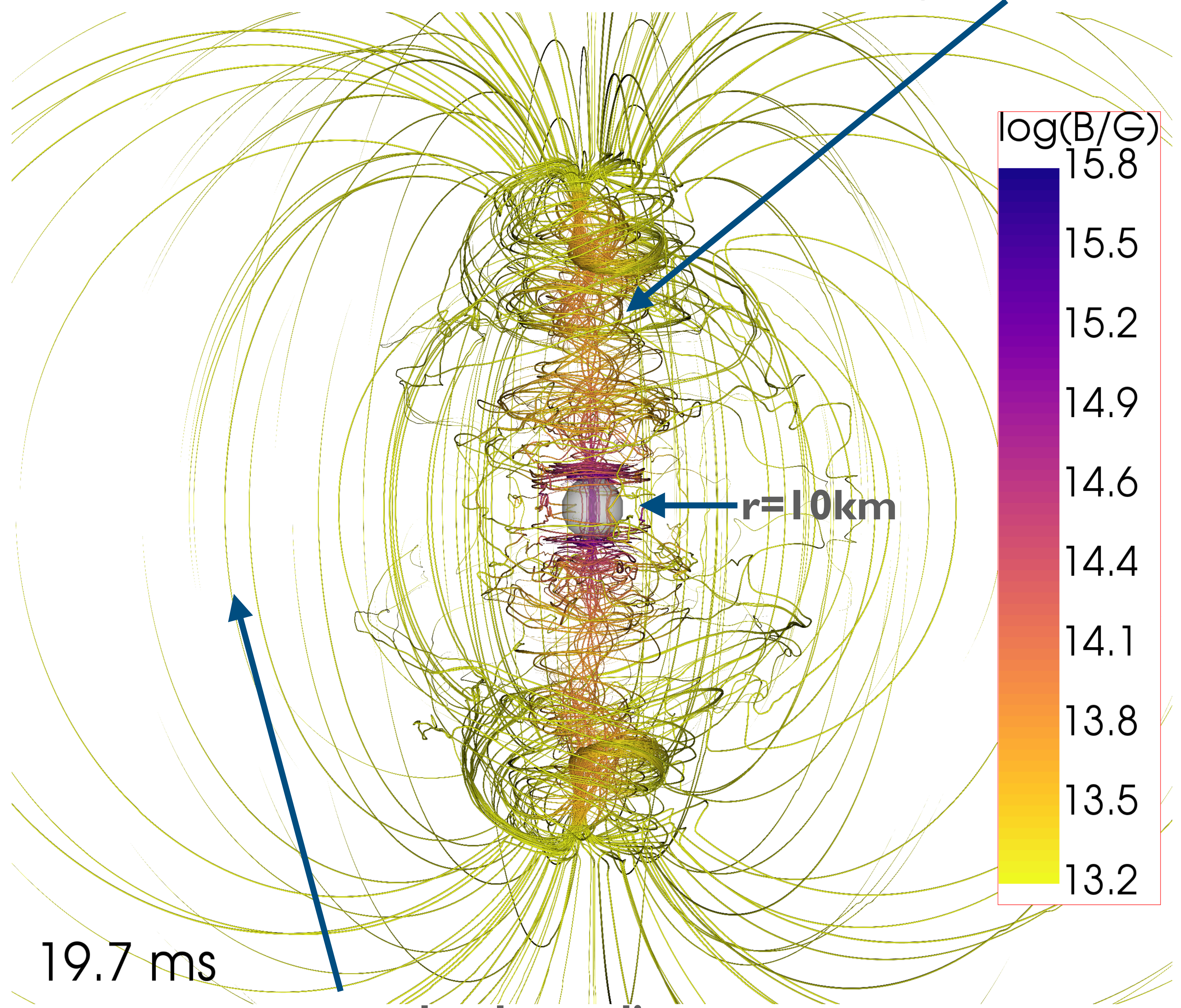
III. Rotating Magnetised Neutron Star

- ID generated using Hydro_RNSID thorn
- $\rho_{\mathrm{c,initial}} = 6.87 \times 10^{14}\,\mathrm{g/cm^3}; \quad \rho_{\mathrm{atm}} = 6.3 \times 10^{5}\,\mathrm{g/cm^3}; \quad B_{\max} \simeq 6.6 \times 10^{15}\,\mathrm{G}; \quad \Omega = 500\,\mathrm{Hz}$
- Field imposed at 10ms; gets slightly distorted due to low density material around the NS
- Differential rotation causes magnetic winding; toroidal component developed

Magnetic field strength



Magnetic winding



Dipolar structure preserved at large distances

Summary

- Kastaun C2P: a robust, accurate and efficient scheme
- Validated for high magnetisations and highly relativistic regimes
- Implementation in Spritz: successfully passed demanding 3D GRMHD tests

Tests in progress:

- Rotating magnetised NS collapse
- Magnetised Fishbone-Moncrief disk

Outlook:

- Fully tabulated EOS
- BNS mergers with Kastaun C2P

Backup Slides

Kastaun C2P Scheme

conservatives

$$D = \rho W = \bar{D}$$

$$\tau = \bar{D}(hW - 1) - P + \frac{1}{2}(E^2 + B^2)$$

$$= \bar{\tau} + \frac{1}{2}(E^2 + B^2)$$

$$S_i = \bar{D}Whv_i + \epsilon_{ijk}E^jB^k = \bar{S}_i + \epsilon_{ijk}E^jB^k$$

useful relations

$$\bar{q} = \frac{\bar{\tau}}{\bar{D}}, \quad q = \frac{\tau}{\bar{D}}, \quad \bar{r}_i = \frac{\bar{S}_i}{\bar{D}}, \quad r_i = \frac{S_i}{\bar{D}}, \quad \tilde{b}_i = \frac{B_i}{\sqrt{\bar{D}}}$$

$$\mu = \frac{1}{\frac{h}{W} + \bar{r}^2\mu}, \quad x \equiv \frac{1}{1 + \mu\tilde{b}^2}, \quad z \equiv vW, \quad a = \frac{P}{\rho_{\mathrm{E}}}$$

$$\bar{r}^2(\mu) = r^2x^2(\mu) + \mu x(\mu)\big(1 + x(\mu)\big)\left(r^l\tilde{b}_l\right)^2$$

$$\bar{q}(\mu) = q - \frac{1}{2}\tilde{b}^2 - \frac{1}{2}\mu^2x^2(\mu)\tilde{b}^2r_\perp^2$$

computed primitives

$$\hat{v}^i(\mu) = \mu\bar{r}^i = \mu x\left(r^i + \mu\left(\tilde{b}^l r_l\right)\tilde{b}^i\right)$$

$$\hat{W}(\mu) = 1/\sqrt{1 - \hat{v}(\mu)^2}$$

$$\hat{\rho}(\mu) = \bar{D}/\hat{W}(\mu)$$

$$\hat{\epsilon}(\mu) = \hat{W}(\mu)\big(\bar{q}(\mu) - \mu\bar{r}^2(\mu)\big) + \hat{v}^2(\mu)\frac{\hat{W}(\mu)^2}{1 + \hat{W}(\mu)}$$

computed using EOS

$$\hat{P}(\mu) = P(\hat{\rho}(\mu), \hat{\epsilon}(\mu))$$

Kastaun C2P Scheme

conservatives

$$D = \rho W = \bar{D}$$

$$\tau = \bar{D}(hW - 1) - P + \frac{1}{2}(E^2 + B^2)$$

$$= \bar{\tau} + \frac{1}{2}(E^2 + B^2)$$

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